On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas

Christina Peraki
Sergio D. Servetto

School of Electrical and Computer Engineering — Cornell University
Outline

● **Goal:** Determination of the *maximum stable throughput* achievable in random networks with directional antennas.
  – Problem Setup.
  – Why Directional Antennas?
  – Some Related Work.

● **Formulation:** Network flow problem in a random unit-disk graph.
  – Flows on Random Graphs.
  – Restriction of Optimization Domain.
  – Max-flow/Min-cut Theorem.

● **Models of Communication:**
  – Omnidirectional Transmissions.
  – “Simple” Directional Transmissions.
  – “Complex” Directional Transmissions.
Outline

• **Goal:** Determination of the *maximum stable throughput* achievable in random networks with directional antennas.
  – Problem Setup.
  – Why Directional Antennas?
  – Some Related Work.

• **Formulation:** Network flow problem in a random unit-disk graph.
  – Flows on Random Graphs.
  – Restriction of Optimization Domain.
  – Max-flow/Min-cut Theorem.

• **Models of Communication:**
  – Omnidirectional Transmissions.
  – “Simple” Directional Transmissions.
  – “Complex” Directional Transmissions.
Problem Setup

- \( n \) nodes placed on \([0, 1] \times [0, 1]\) uniformly at random.
- Each node directly connected to nodes within distance \( d_n \).
- Links: fixed finite capacity \( L \) independent of network size.

- Find MST: total number of packets all sources can inject into the network by keeping the size of the largest queue bounded.
Why directional antennas?

- Higher degree of spatial reuse of shared medium.
- Smaller number of hops visited by a packet on its way to its destination.
- Can their use provide gains in MST?
Some Related Work

- Condition for the graph to be connected with probability 1, as $n \to \infty$, and for any $\xi_n \to \infty$:

$$\pi d_n^2 = \frac{\log n + \xi_n}{n},$$


- Total throughput: $\Theta\left(\sqrt{n/\log n}\right)$—per node: $\Theta\left(1/\sqrt{n \log(n)}\right)$.


Can directional antennas be used to achieve non-vanishing throughput per node? If so, at what cost?
Outline

- **Goal:** Determination of the maximum stable throughput achievable in random networks with directional antennas.
  - Problem Setup.
  - Why Directional Antennas?
  - Some Related Work.

- **Formulation:** Network flow problem in a random unit-disk graph
  - Flows on Random Graphs.
  - Restriction of Optimization Domain.
  - Max-flow/Min-cut Theorem.

- **Models of Communication:**
  - Omnidirectional Transmissions.
  - “Simple” Directional Transmissions.
  - “Complex” Directional Transmissions.

---

On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas. ACM MobiHoc—6/02/03.
Flows on Random Graphs

Finding maximum stable throughput in our network: instance of multicommodity flow problem:

- $n$ commodities: Packets from source $s_i$ to receiver $t_i$.
- Sum of packets transmitted by all sources cannot exceed capacity of a link.
- What is the largest number of packets that can be injected simultaneously by all sources?

Whenever possible, solvable using linear programming methods.

---

*On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas.* ACM MobiHoc—6/02/03.
Restriction of Optimization Domain

Not much is known about the **structure** of optimal solutions to the maximum multicommodity flow problem. Special case of high interest:

- Still have constriction in capacity.
- Can make use of standard flow methods to solve the problem.
- Scaling laws obtained coincide with those in the Gupta-Kumar capacity paper: *this restriction does not change the rate of growth of the value of the linear programs.*
Max-flow/Min-cut Theorem

- The value of any flow $f$ from the supersource $s$ to the supersink $t$ in $G$ is bounded from above by the capacity of any cut of $G$ for which $s \in S$ and $t \in T$.

- According to the max-flow/min-cut theorem, $f$ is a flow of maximum value iff $|f| = c(S, T)$ (for some cut $(S, T)$).

We determine how many edges straddle simultaneously a minimum cut in order to compute the maximum flow.

On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas. ACM MobiHoc—6/02/03.
Proof Technique: Two Steps

- First, compute the **expected** number of edges that straddle the central cut.

  *Determination of ensemble averages.*

- Then prove that, in almost all networks, the **actual** number of edges that straddle the central cut differs from the mean at most by a constant factor.

  *Proof of sharp concentration around the mean.*
Tools

- Expected number of points in a set $A \subset [0, 1] \times [0, 1]$: If $N$ the number of points in $A$ then:

$$E(N) = n|A|,$$

where $n$ the total number of points in $[0, 1] \times [0, 1]$.

- Chernoff bounds for sharp concentration results:

$$P \left[ |N - n|A|| > \delta n|A| \right] < e^{-\theta n|A|},$$

for any $0 < \delta < 1$, $\theta$ a function of $\delta$ and $\theta > 0$ always.
Outline

- **Goal**: Determination of the *maximum stable throughput* achievable in random networks with directional antennas.
  - Problem Setup.
  - Why Directional Antennas?
  - Some Related Work.

- **Formulation**: Network flow problem in a random unit-disk graph.
  - Flows on Random Graphs.
  - Restriction of Optimization Domain.
  - Max-flow/Min-cut Theorem.

- **Models of Communication**:
  - Omnidirectional Transmissions.
  - “Simple” Directional Transmissions.
  - “Complex” Directional Transmissions.
OMNIDIRECTIONAL ANTENNAS (I)

- For a transmission to be successfully decoded, only one transmission has to be in progress within the range of a receiver.

A transmission model based on omnidirectional antennas and pure collisions.

On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas. ACM MobiHoc—6/02/03.
OMNIDIRECTIONAL ANTENNAS (II)

- Average number of edges across the cut:

- For a receiver at location \((x, y)\), at most one transmitter in the shaded region \(T_{xy}\) can send a message. An upper bound to the expected number of transmissions is:

\[
\frac{2}{\pi d_n} = O \left( \sqrt{\frac{n}{\log n}} \right).
\]
OMNIDIRECTIONAL ANTENNAS (III)

- Upper bound is asymptotically tight. Explicit flow construction:

\[ \lim_{n \to \infty} P \left[ \bigcap_{j=1}^{1/2d_n} j\text{-th circle has } \Theta(\log(n)) \text{ nodes} \right] = 1. \]

- A uniform convergence result:

  MST is \( \Theta \left( \frac{1}{2d_n} \right) = \Theta \left( \frac{2}{\pi d_n} \right) = \Theta \left( \sqrt{n \log(n)} \right) \), matching known laws.


On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas. ACM MobiHoc—6/02/03.
DIRECTIONAL ANTENNAS

- Change of collision model:
  - Transmitters can generate beams of arbitrarily narrow width.
  - Receivers can resolve different transmissions as long as there is an arbitrarily small positive angle between receptions.

- This model presents the most favorable set of assumptions on directional antennas...

---

On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas. ACM MobiHoc—6/02/03.
A SINGLE DIRECTED BEAM (I)

- Transmitters can generate a beam of arbitrarily narrow width aimed at any particular receiver.
- Receivers can accept any number of incoming messages, provided transmitters are not in same straight line.

A single beam model for communication between nodes.
A SINGLE DIRECTED BEAM (II)

- Average number of edges across the cut:

- Average number of transmitters in $L$ and receivers in $R$ is $nd_n$.

- Maximum of $nd_n(1 - \frac{1}{e})$ transmissions can actually be received on average—*a standard occupancy problem of throwing balls into bins.*


- Using the Chernoff Bound again, MST is $\Theta(nd_n)$.
A SINGLE DIRECTED BEAM (III)

- Number of edges across the cut is $\Theta(nd_n) = \Theta\left(\sqrt{n \log(n)}\right)$.

- Increase from omnidirectional case: $\Theta(\log(n))$.

- Why don’t we increase the connectivity radius to get linear MST?
  - In that case $d_n$ should satisfy: $\Theta(nd_n) = \Theta(n)$, and so $d_n = \Theta(1)$.
  - Minimum number of resolvable beams required:
    $$\gamma = n \cdot \pi d_n^2 = n \cdot \Theta(1)^2 = \Theta(n),$$
    maximum angle of dispersion decays linearly in $n$, and exponentially in $\log(n)$ (minimum connectivity requirement).
MULTIPLE DIRECTED BEAMS (I)

- Transmitters can generate an arbitrary number of beams, of arbitrarily narrow width, aimed at any particular receiver.

- Receivers can accept any number of incoming messages, provided the transmitters are not in the same straight line.
MULTIPLE DIRECTED BEAMS (II)

- Average number of edges across the cut:

- For an arbitrary point in $L$ the points in $Q_{xy}$ correspond to possible receivers. Adding up for all transmitters: $n^2 \int_L |Q_p| dp = \frac{2}{3} n^2 d_n^3$.

- Maximum of $\frac{2}{3} n^2 d_n^3 (1 - \frac{1}{e})$ transmissions can actually be received on average—same occupancy problem as for the single beam case.

- Using the Chernoff Bound again, MST is $\Theta \left( n^2 d_n^3 \right)$. 

---

On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas. ACM MobiHoc—6/02/03.
MULTIPLE DIRECTED BEAMS (III)

- Number of edges across the cut is $\Theta \left( n^2 d_n^3 \right) = \Theta \left( \sqrt{n \log^2(n)} \right)$.

- Increase from omnidirectional case: $\Theta(\log^2(n))$.

- Why don’t we increase the connectivity radius to get linear MST?
  - In that case $d_n$ should satisfy: $\Theta \left( n^2 d_n^3 \right) = \Theta(n)$, so $d_n = \Theta(n^{-\frac{1}{3}})$.
  - Minimum number of resolvable beams required:
    $$\gamma = n \cdot \pi d_n^2 = n \Theta(n^{-\frac{2}{3}}) = \Theta(n^{\frac{1}{3}}),$$
    maximum angle of dispersion still decays polynomially in $n$, and exponentially in $\log(n)$ (minimum connectivity requirement).
Final Remarks

Summary:

- Analyzed the rate of growth for MST in various tx/rx architectures.
- Formulated the problem as a maximum flow one in random graphs.

Conclusions:

- Under very ideal assumptions, can gain at most $\Theta(\log^2(n))$ in MST by using directional antennas.
- Directional antennas are not expected to provide meaningful throughput gains in practice ... but can maybe help in other ways?

Current work: prove equivalence of linear programs, and develop numerical simulations.

On the Maximum Stable Throughput Problem in Random Networks with Directional Antennas. ACM MobiHoc—6/02/03.