# A Quick Summary on Probability for ECE446

(Compiled by Ron.)

**Abstract**

*Part I is a summary of the basic tools I expect you to remember from your previous probability class. Part II is a sketchy example of the kind of uses we will have for the background material. In the lecture on 8/30 we will look at some of these examples in more detail. In the background quiz, I will NOT test you on how to use these ideas applied to concrete networks, that's the topic for this class – instead, I will give you 310-like type of problems related to Section I. Happy reviewing... :-)*

## I. Probability Distributions

*Definition 1.1:* We say that two random variables $X_1$ and $X_2$ are *independent* if and only if the joint distribution $\Pr(X_1 = x_1, X_2 = x_2) = \Pr(X_1 = x_1)\Pr(X_2 = x_2)$.

*Definition 1.2:* Let $X_1, X_2, ...X_n$ be a set of $n$ random variables. We say that these variables are *independent and identically distributed* (i.i.d) if the joint distribution $\Pr(X_1 = x_1, X_2 = x_2, ..., X_n = x_n) \triangleq \Pr(x_1, x_2, ..., x_2) = \prod_{i=1}^{n} \Pr(X_i = x_i) \triangleq \prod_{i=1}^{n} \Pr(x_i)$

*Definition 1.3:* A *Bernoulli* random variable (RV) $X$ with parameter $p$ is defined by the following distribution:

$$\Pr(X = 0) = 1 - p, \quad \Pr(X = 1) = p.$$

We denote this $X \sim \text{Bern}(p)$.

Let $X_1, X_2, ..., X_n$ be i.i.d. RVs with $X_i \sim \text{Bern}(p)$. Let $S_n = \sum_{i=1}^{n} X_i$.

Examples:

- $n$ coin tosses. $S_n$ counts the number of $1$ in these trials.
- $n$ packets are sent, $X_i = 1$ indicates successful reception. $S_n$ counts the number of packets received successfully.

Applying the definition of i.i.d. variables in definition 1.2 we can write

$$\begin{aligned}
\Pr\left(X_1 = 0, X_2 = 1, X_3 = 1\right) &= \Pr(X_1 = 0)\Pr(X_2 = 1)\Pr(X_3 = 1) \\
&= (1 - p) \cdot p \cdot p \\
&= p^2(1 - p).
\end{aligned}$$

Similarly

$$\begin{aligned}\Pr\left(X_1 = 1, X_2 = 1, X_3 = 0\right) &= p^2(1-p)\\ \Pr\left(X_1 = 1, X_2 = 0, X_3 = 1\right) &= p^2(1-p)\end{aligned}$$

Therefore

$$\begin{aligned}\Pr(S_3) &= \Pr\left((X_1 = 0, X_2 = 1, X_3 = 1) \text{ or } (X_1 = 1, X_2 = 1, X_3 = 0) \text{ or } (X_1 = 1, X_2 = 0, X_3 = 1)\right)\\ &= \Pr\left(X_1 = 0, X_2 = 1, X_3 = 1\right) + \Pr\left(X_1 = 1, X_2 = 1, X_3 = 0\right) + \Pr\left(X_1 = 1, X_2 = 0, X_3 = 1\right)\\ &= 3p^2(1-p).\end{aligned}$$

As can be seen, the order of the $1$'s and $0$'s does not matter. Therefore we have

$$\Pr(S_n = k) = (\text{\# of sets with } k \text{ 1's and } n-k \text{ 0's}) \cdot p^k(1-p)^{n-k}$$

*Definition 1.4: Permutations*: The number of ways to arrange $n$ distinct objects on a line.
The number of permutations for $n$ distinct objects is $n!$

*Definition 1.5: Combinations*: The number of ways to select $k$ distinct objects out of $n$ distinct objects when the selection order does not matter. This number is denoted by $C(n,k)$.
Note that $C(n,k) = \frac{n!}{k!(n-k)!}$. To see this, assume that the first $k$ items in the arrangement of the $n$ items are selected. Then we have $n!$ distinct arrangements which accounts to the numerator. But the order of selection is not important so we divide by $k!$. Also the order of the remaining elements is not important hence we divide again by $(n-k)!$.
Now we can define the following

*Definition 1.6:* A Binomial RV $X$ is defined by the probability

$$\Pr\left(X = k\right) = C(n,k)p^k(1-p)^{n-k}.$$

We denote this with $X \sim B(n,p)$. Not that the possible values of $X$ range from $0$ to $n$.
Clearly, we have that $S_n \sim B(n,p)$.

*Definition 1.7:* The *expectation* (also referred to as mean) of a random variable is defined by

$$E(X) = \sum_{x\in\Omega} x \Pr(X = x)$$

where $\Omega$ is the sample space of $X$ (namely the set of possible values $X$ can assume).

- The expectation is linear $E(X+Y) = E(X) + E(Y)$.
- For a constant number $a$ we have that $E(aX) = aE(X)$.

Example:

Mean for a binomial RV $S_n$:

$$\begin{aligned} E(S_n) &= E\left(\sum_{i=1}^{n} X_i\right) \\ &\overset{(a)}{=} \sum_{i=1}^{n} E(X_i) \\ &= \sum_{i=1}^{n} (1 \cdot p + 0 \cdot (1-p)) \\ &= \sum_{i=1}^{n} p \\ &= np \end{aligned}$$

where in (a) we used linearity of expectation.

*Comment 1.1:* Note that contrary to addition, for multiplication in general

$$E(X_1 \cdot X_2) \neq E(X_1)E(X_2).$$

However, for **independent** random variables we do have $E(X_1 \cdot X_2) = E(X_1)E(X_2)$

Now consider another interesting question: what is the probability of receiving a run of $k-1$ zeros followed by a single $1$ (probability that the first success is in the $k$'th transmission), $\Pr(X_1 = 0, X_2 = 0, ..., X_{k-1} = 0, X_k = 1)$?

Using the i.i.d. property we have that

$$\begin{aligned} \Pr(X_1 = 0, X_2 = 0, ..., X_{k-1} = 0, X_k = 1) &= \prod_{i=1}^{k-1} \Pr(X_i = 0) \Pr(X_k = 1) \\ &= (1-p)^{k-1} p \end{aligned}$$

*Definition 1.8:* A random variable $X$ is said to have a $Geometric$ distribution with parameter $p$ if

$$\Pr(X = k) = (1-p)^{k-1} p,$$

where $\Omega = 1, 2, ...$ . We denote this with $X \sim G(p)$.

The mean for RV $X$ which follows a Geometric distribution can be calculated as

$$\begin{aligned}
E(X) &= \sum_{k=1}^{\infty} k \Pr(X=k) \\
&= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\
&= -p\frac{\partial}{\partial p}\sum_{k=0}^{\infty}(1-p)^k \\
&= -p\frac{\partial}{\partial p}\frac{1}{p} \\
&= p\frac{1}{p^2} \\
&= \frac{1}{p}
\end{aligned}$$

Lastly we define a uniform RV:

*Definition 1.9:* We say that a RV $X$ has a *uniform* distribution with parameters $a$ and $b$, denoted $X \sim U(a,b)$ if

$$\Pr(X=i) = \frac{1}{b-a},$$

where $\Omega = a, a+1, ..., b$, and $b > a$.

## II. Network Performance

Consider a point-to-point network, with a source $S$ and destination $T$.

*Definition 2.1:* The *throughput* of the network is defined as the average packets per unit time that can be transmitted from $S$ to $T$ (when using the maximum possible rate):

$$\mu \triangleq \frac{\text{average \# of packets}}{\text{unit of time}}$$

Note that $\mu$ depends on the network characteristics (probability of losing a packet, delays in the network) as well as on the protocol governing the transmission from $S$ to $T$.

Example 1:

Consider a source transmitting over a perfect channel (no packets are loss) where before the next packet is transmitted the source must receive acknowledgement on the transmission of the previous packet. Assume the roundtrip delay is $t_D$. The throughput of this network is trivially $\frac{1}{t_D}$. From this simple example we can see a basic characteristic of the network: smaller delays imply higher throughputs.

Example 2:

Consider now a network with transmitter $S$ and receiver $T$, where the packets are send in groups of $N$ packets, all sent in the same instant, and the roundtrip time is $t_D$. However, the probability of successful reception at $T$ is $p$, and we assume that the ACK messages from the receiver $T$ are received at the transmitter $S$ without error. Now, for finding the throughput we need first the average rate of successful packet reception. Clearly we can identify the number of successfully received packets as a Binomial RV $B(N, p)$. Therefore its expectation, as computed above is $Np$. Now we have to divide it but the overall time it takes to transmit this over the network which is $t_D$. We get

$$\mu = \frac{Np}{t_D}.$$